Math 115A, Lecture 2
Linear Algebra

Midterm 2

Instructions: You have 50 minutes to complete the exam. There are five problems, worth a total of fifty points. You may not use any books or notes. Partial credit will be given for progress toward correct proofs.

Write your solutions in the space below the questions. If you need more space use the back of the page. Do not forget to write your name in the space below.

Name: ____

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

Problem 1.

Let $T : V \to W$ be a linear transformation between real vector spaces V and W.

(a) [5pts.] Define $R(T)$ and $N(T)$.

> **Solution:** The range $R(T)$ is the subspace $\{T(v) : v \in V\} \subset W$. The null space $N(T)$ is the subspace $\{v \in V : T(v) = 0\}$.

(b) [5pts.] Prove that T is one-to-one if and only if $N(T) = \{0\}$.

> **Solution:** For clarity, let 0_V be the identity element in V and 0_W be the identity element in W. First, suppose that T is one-to-one. Then if $T(v) = 0_W = T(0_V)$, we must have $v = 0_V$. Ergo $N(T) = \{0_V\}$. Now, conversely, suppose that $N(T) = \{0_V\}$. Let v_1, v_2 be two vectors in V such that $T(v_1) = T(v_2)$. Then by linearity, $T(v_1 - v_2) = T(v_1) - T(v_2) = 0_W$, so $v_1 - v_2 \in N(T)$. But this implies that $v_1 - v_2 =)_V$, or in other words $v_1 = v_2$. So we have proved that $T(v_1) = T(v_2)$ only if $v_1 = v_2$, we see that T is one-to-one.

Problem 2.

Consider the linear transformation

$$\begin{aligned} T : P(\mathbb{R}) &\to P(\mathbb{R}) \\ f(x) &\mapsto f'(x) + f(0) \end{aligned}$$

(a) [3pts.] Is T onto?

> **Solution:** Yes. It suffices to show that every element of the basis $\{1, x, x^2, \cdots\}$ for $P(\mathbb{R})$ is in $R(T)$. But we observe that $x^n = T(\frac{1}{n+1}x^n)$ for all $n \geq 0$. Ergo T is onto.

(b) [3pts.] Is T an isomorphism?

> **Solution:** No, T is not one-to-one. Notice that $T(x - 1) = 1 - 1 = 0$. (Notice that the dimension theorem does not apply here, since $P(\mathbb{R})$ is infinite dimensional.

(c) [4pts.] What are the eigenvectors of T?

> **Solution:** Notice that if $v = c_0 + c_1x + \cdots + c_nx^n \in P(\mathbb{R})$, then $T(c_0 + c_1x + \cdots + c_nx^n) = (c_0 + c_1) + 2c_2x + \cdots + nc_nx^{n-1}$. We see that the constant polynomials are eigenvectors of T with eigenvalue $\lambda = 1$, and the polynomials $f(x) = c - cx$ are eigenvectors with eigenvalue $\lambda = 0$.

Problem 3.

Let T be the linear transformation of the plane given by rotating $\frac{\pi}{4}$ radians counterclockwise and then reflecting across the x-axis.

(a) [5pts.] Find the matrix representing $[T]_\beta$ representing T with respect to the standard basis β for $\mathbb{R}^2$.

> **Solution:** Rotation by $\frac{\pi}{4}$ maps $(1,0)$ to $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and $(0,1)$ to $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, whereas reflection across the x-axis maps $(1,0)$ to itself and $(0,1)$ to $(0,-1)$. Therefore we conclude that
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> $$[T]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

(b) [5pts.] Find a basis β' for $\mathbb{R}^2$ with respect to which T is represented by a diagonal matrix.

> **Solution:** We see that the characteristic polynomial of $[T]_\beta$ is $\frac{1}{2}(t^2 - 1)$, so the eigenvalues of T are ± 1. Solving $[T]_\beta[v] = \pm v$ shows that $\beta' = \{(1, 1 - \sqrt{2}), (1, 1 + \sqrt{2})\}$ is a basis for $\mathbb{R}^2$ consisting of eigenvectors for T. With respect to this basis, we have
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> $$[T]_{\beta'} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Problem 4.

Let V be a finite-dimensional vector spaces, and let $\mathcal{L}(V)$ be the vector space of linear transformations from V to itself.

(a) [5pts.] Consider the subset Z of $\mathcal{L}(V)$ consisting of the invertible linear transformations from V to itself. Is Z a subspace of $\mathcal{L}(V)$?

> **Solution:** No. For the easiest reason, we see that the zero transformation T_0, which is the identity element of $\mathcal{L}(V)$ is not in Z. More generally, Z is never closed under addition.

(b) [5pts.] Show that $\beta = \{T_1, \cdots, T_n\}$ is a basis for $\mathcal{L}(V)$ and $S \in \mathcal{L}(V)$ is invertible, then $S(\beta) = \{ST_1, \cdots, ST_n\}$ is also a basis for $\mathcal{L}(V, W)$. [Hint: You will want to use the fact that S^{-1} exists.]

> **Solution:** We must show that $\{ST_1, \cdots, ST_n\}$ is linearly independent and generates V. First, suppose there is some linear relationship $a_1ST_1 + \cdots + a_nST_n = 0$. Then we have $S(aT_1 + \cdots + aT_n) = T_0$, since composition distributes over

addition and scalar multiplication. Applying S^{-1} to both sides of this equality gives $aT_1 + \cdots + a_nT_n = S^{-1}T_0 = T_0$, so since β is linearly independent, $a_1 = \cdots = a_n = 0$. We conclude that $S(\beta)$ is linearly independent.

Next, we claim that $S(\beta)$ generates $\mathcal{L}(V, W)$. For let $U \in \mathcal{L}(V, W)$. Then $S^{-1}U = a_1T_1 + \cdots + a_nT_n$ for some $a_1, a_2, \cdots, a_n \in \mathbb{R}$, so applying S to both sides of this equality gives $U = a_1ST_1 + \cdots + a_nST_n$. We conclude that $S(\beta)$ generates $\mathcal{L}(V, W)$.

Problem 5.

Let $T : V \to V$ be a linear transformation.

(a) [5pts.] Prove that if λ is an eigenvalue of T, then λ^n is an eigenvalue of T^n.

Solution: Suppose that $v \in V$ is an eigenvector of T with eigenvalue λ. Then $T(v) = \lambda v$. We claim that $T^n(v) = \lambda^n(v)$. We will show this inductively. The base case $n = 1$ is true by assumption. For the inductive step, if we assume that $T^{n-1}(v) = \lambda^{n-1}v$, then $T^n(v) = T(T^{n-1}(v)) = T(\lambda^{n-1}(v)) = \lambda^{n-1}T(v) = \lambda^{n-1}(\lambda(v)) = \lambda^n v$.

(b) [5pts.] Suppose that $\lambda > 0$ is an eigenvalue of T^n. Is $\lambda^{\frac{1}{n}}$ necessarily an eigenvalue of T? [Hint: Think about rotations.]

Solution: No. Consider the rotation $T_{\frac{\pi}{2}}$ by $\frac{\pi}{2}$ radians in the plane, which we observed in class has no eigenvalues. The fourth power $T_{\frac{\pi}{2}}^4 = T_{2\pi} = I_{\mathbb{R}^2}$. But of course every nonzero vector in $\mathbb{R}^2$ is an eigenvector of the identity transformation with eigenvalue $\lambda = 1$.